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On fractional Duhamel's principle and its applications

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ABSTRACT

The classical Duhamel principle, established nearly 200 years ago by Jean-Marie-Constant Duhamel, reduces the Cauchy problem for an inhomogeneous partial differential equation to the Cauchy problem for the corresponding homogeneous equation. In this paper we generalize this famous principle to a wide class of fractional order differential-operator equations.

1. Introduction

Let X be a reflexive Banach space and $A:\mathcal{D}\to X$ be a closed linear operator with a domain $\mathcal{D}\subset X$. In Section 2 we will introduce a Frechét type topological vector space $\mathrm{Exp}_{A,G}(\mathrm{X})$ and its dual, where G is an open subset of the complex plain $\mathbb{C}$. This space represents a modification of the space of entire functions with finite exponential type [8,10,23,28] and its abstract versions. We also introduce a functional calculus $f(A)$, where f is an analytic function defined on G. The function f is called the symbol of the operator $f(A)$.

The goal of this paper is to generalize Duhamel's principle for the Cauchy problem to general inhomogeneous fractional distributed order differential-operator equations of the form

$$L^{\Lambda}[u]\equiv\int_0^{\mu} f(\alpha,A)D_*^{\alpha}u(t)\,d\Lambda(\alpha)=h(t),\quad t>0, \tag{1}$$

$$u^{(k)}(0)=\varphi_k,\quad k=0,\dots,m-1, \tag{2}$$

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where $\mu \in (m-1, m]$ and $h(t)$ and φ_k, $k = 0, \dots, m-1$, are given X-valued vector-functions. The family of operators $f(\alpha, A)$ is defined through the family of symbols $f(\alpha, z)$ that are continuous in the variable $\alpha \in [0, \mu]$, and analytic in the variable $z \in G \subset \mathbb{C}$. The measure Λ is finite and defined on $[0, \mu]$. Finally, D_*^α denotes the operator of fractional differentiation of order α in the sense of Caputo and Djrbashian (see, for example, [5,18]), i.e.

$$D_*^\alpha g(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{g^{(n)}(\tau)\, d\tau}{(t-\tau)^{\alpha-n+1}}, & \text{if } n-1 < \alpha < n,\ n \in \mathbb{N}, \\ g^{(n)}(t) \equiv \frac{d^n}{dt^n} g(t), & \text{if } \alpha = n \in \{0\} \cup \mathbb{N}. \end{cases} \tag{3}$$

Hereafter the integrals are understood in the sense of Bochner if $g(t)$ is a vector-function with values in some topological vector space for each fixed t.

In the particular case of fractional order partial differential equations with a single "fractional" term in Eq. (1), a fractional analog of Duhamel's principle was obtained in [36,37]. Unlike the classical case, the fractional Duhamel principle requires a fractional derivative of $h(t)$ in the initial condition of the reduced Cauchy problem. In this paper we formulate and prove *fractional generalizations of Duhamel's principle* applicable directly to the Cauchy problem for inhomogeneous fractional order differential-operator equations, which reduce them to the Cauchy problem for the corresponding homogeneous equations. The obtained results are new even in the scalar case.

Fractional order differential equations are useful for modeling problems with memory, and interest in this subject has grown substantially during the last few decades. For instance, probability density functions of a wide class of non-Gaussian diffusion processes satisfy space and time fractional order differential equations (see [1,2,17,26] and references therein). Inhomogeneous fractional order differential equations appear naturally as a description of the influence of an external force or memory effects. In particular, in the study of diffusion processes in complex heterogeneous media with several distinct diffusion modes, even without an external force, the function $h(t)$ emerges and embodies memory of the past [16,24,38].

There is an extensive literature on the Cauchy problem for integer order abstract differential-operator equations (see, e.g. [22,41]). The first order evolution equations $u'(t) = Au(t)$ in the spaces of abstract exponential vector-functions of a finite type, $\mathrm{Exp}_A(X)$ (and in more general bornological spaces) were studied in [28]. For integer order pseudo-differential and differential-operator equations with analytic symbols or with symbols having singularities the Cauchy and multi-point problems were studied, for example, in [9,39,27,32–34]. What regards to fractional order differential-operator equations, Kochubei [19] studied existence and uniqueness of a solution to the abstract Cauchy problem $D_*^\alpha u(t) = Au(t)$, $u(0) = u_0$, with Caputo–Djrbashian fractional derivative for $0 < \alpha < 1$ and a closed operator A with a dense domain $\mathcal{D}(A)$ in a Banach space. El-Sayed [7] and Bazhlekova [3] investigated Cauchy problem for $0 < \alpha < 2$. In the general case of $\alpha > 0$, Gorenflo et al. [15] studied existence of solutions in Roumieu–Beurling and Gevrey classes. Kostin [21] proved that the abstract initial value problem (Cauchy type problem) $D_+^\alpha u(t) = Au(t)$, $D_+^{\alpha-k} u(0) = \varphi_k$, $k = 1, \dots, m$, where $\alpha \in (m-1, m)$ and D_+^α is the Riemann–Liouville derivative, is well posed. For more information about recent results on the Cauchy problem for abstract fractional differential-operator equations, we refer the reader to [3,6,18]; and for a recent mathematical treatment of the distributed fractional order differential equations to papers [20,25,35].

This paper is organized as follows. In Section 2 we recall the classic Duhamel principle, and the basic reference vector spaces used in this paper. Since we formulate a fractional Duhamel principle, we introduce a topological vector space on which the corresponding operators act. In Section 3 we formulate the main result, namely fractional analog of Duhamel's principle and discuss some of its applications.

2. Preliminaries

2.1. Fractional order derivatives

For a function g defined on $[\tau,\infty)$, under the condition of absolute integrability the *fractional integral of order* $\beta>0$ with *terminal points* τ and t, is defined as [30]

$$ {}_\tau J^\beta g(t)=\frac{1}{\Gamma(\beta)}\int_\tau^t (t-s)^{\beta-1}g(s)\,ds,\quad t>\tau, $$

where $\Gamma(\cdot)$ is Euler's gamma-function. Obviously, if $\beta=n$, a natural number, then ${}_\tau J^n$ is the n-fold integral of f over the interval $[\tau,t]$. By convention, ${}_\tau J^0 f(t)=f(t)$, i.e. ${}_\tau J^0$ coincides with the identity operator.

Furthermore, let m be a positive integer number. We denote by ${}_\tau D_+^\alpha$ for $m-1<\alpha<m$, the fractional derivative of order α in the sense of Riemann and Liouville, which is defined as [30]

$$ {}_\tau D_+^\alpha g(t)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_\tau^t \frac{g(s)\,ds}{(t-s)^{\alpha+1-m}},\quad t>\tau, $$

and ${}_\tau D_+^0 g(t)=g(t)$, ${}_\tau D_+^m g(t)=g^{(m)}(t)$. Between this fractional derivative and the *Caputo–Djrbashian* derivative given in (3) the following relationship holds [18]:

$$ {}_\tau D_+^\alpha g(t)={}_\tau D_*^\alpha g(t)+\sum_{k=0}^{m-1}\frac{g^{(k)}(\tau)}{\Gamma(k-\alpha+1)}(t-\tau)^{k-\alpha},\quad t>\tau. \tag{4} $$

In the particular case of $0<\alpha<1$ one has

$$ {}_\tau D_+^\alpha g(t)={}_\tau D_*^\alpha g(t)+g(\tau)\frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)},\quad t>\tau. \tag{5} $$

If $g(\tau)=0$, then one obtains the equality ${}_\tau D_+^\alpha g(t)={}_\tau D_*^\alpha g(t)$. To clarify,

$$ {}_\tau D_+^\alpha g(t)=\frac{d^m}{dt^m}\,{}_\tau J^{m-\alpha}g(t)\quad\text{and}\quad {}_\tau D_*^\alpha g(t)={}_\tau J^{m-\alpha}\frac{d^m}{dt^m}g(t). $$

We omit from the notation the lower terminal point τ if $\tau=0$, writing simply D_+^α, D_*^α or J^α. Recall that for the Laplace transform of $D_*^\alpha g(t)$, where $\alpha\in(m-1,m]$ and g is m times differentiable, Laplace transformable, and $D^k g(t)\to 0$, $k=0,\dots,m-1$, as $t\to\infty$, the following formula is valid [18]:

$$ \mathcal{L}\big[D_*^\alpha g\big](s)=s^\alpha\mathcal{L}[g](s)-\sum_{k=0}^{m-1}g^{(k)}(0+)s^{\alpha-1-k}. \tag{6} $$

Here $\mathcal{L}[g](s)$ denotes the Laplace transform of g.

2.2. An operator calculus

In this section we recall some necessary facts about abstract spaces of analytic elements of finite exponential type, and an operator calculus defined on it. See for details [32,33].

Let X be a reflexive Banach space with a norm $\|v\|$, $v \in X$. Let A be a closed linear operator with a domain $\mathcal{D}(A)$ dense in X and a spectrum $\sigma(A) \subset \mathbb{C}$ being not empty.

We will develop an operator calculus $f(A)$ for analytic functions $f(\lambda)$ in an open domain $G \subset \mathbb{C}$. If G contains $\sigma(A)$ then we define

$$f(A) = \int_{\nu} \mathcal{R}(\zeta, A) f(\zeta)\, d\zeta, \tag{7}$$

where ν is a contour in G containing $\sigma(A)$, and $\mathcal{R}(\zeta, \mathcal{A})$, $\zeta \in \mathbb{C} \setminus \sigma(A)$, is the resolvent operator of A.

In the case when f has singular points in the spectrum $f(A)$ can be constructed as follows. Assume that G is an open set in $\mathbb{C}$ not necessarily containing $\sigma(A)$. Further, let $0 < r \leqslant +\infty$ and $\nu < r$. Denote by $\mathrm{Exp}_{A,\nu}(\mathrm{X})$ the set of elements $v \in \bigcap_{k \geqslant 1} \mathcal{D}(A^k)$ satisfying the inequalities $\|A^k v\| \leqslant C \nu^k \|v\|$ for all $k = 1, 2, \dots,$ with a constant $C > 0$ not depending on k. An element $v \in \mathrm{Exp}_{A,\nu}(X)$ is said to be a vector of exponential type ν [28]. A sequence of elements v_n, $n = 1, 2, \dots,$ is said to converge to an element $v_0 \in \mathrm{Exp}_{A,\nu}(\mathrm{X})$ iff:

1) All the vectors v_n are of exponential type $\nu < r$, and
2) $\|v_n - v_0\| \to 0$, $n \to \infty$.

Obviously, $\mathrm{Exp}_{A,\nu_1}(\mathrm{X}) \subset \mathrm{Exp}_{A,\nu_2}(\mathrm{X})$, if $\nu_1 < \nu_2$. Let $\mathrm{Exp}_{A,r}(\mathrm{X})$ be the inductive limit of spaces $\mathrm{Exp}_{A,\nu}(\mathrm{X})$ when $\nu \to r$. For basic notions of topological vector spaces including inductive and projective limits we refer the reader to [29]. Set $A_\lambda = A - \lambda I$, where $\lambda \in G$, and denote $\mathrm{Exp}_{A,r,\lambda}(\mathrm{X}) = \{u_\lambda \in X\colon\ u_\lambda \in \mathrm{Exp}_{A_\lambda,r}(\mathrm{X})\}$, with the induced topology. Finally, for arbitrary $G \subset \sigma(A)$, denote by $\mathrm{Exp}_{A,G}(\mathrm{X})$ the space whose elements are the locally finite sums of elements in $\mathrm{Exp}_{A,r,\lambda}(\mathrm{X})$, $\lambda \in G$, $r < dist(\lambda, \partial G)$, with the corresponding topology. Namely, any $u \in \mathrm{Exp}_{A,G}(\mathrm{X})$, by definition, has a representation $u = \sum_\lambda u_\lambda$ with a finite sum. It is clear, that $\mathrm{Exp}_{A,G}(\mathrm{X})$ is a subspace of the space of vectors of exponential type if $r < +\infty$, and coincides with it if $r = +\infty$. $\mathrm{Exp}_{A,G}(\mathrm{X})$ is an abstract analog of the space $\Psi_{G,p}(R^1)$ introduced in [32], where $A = -i\frac{d}{dx}$, $G \subseteq R^1$, $\mathrm{X} = L_p(R^1)$, $1 < p < \infty$. In the case $A = -i\frac{d}{dx}$, $\mathrm{X} = L_2(R^1)$, the corresponding space was studied in [9].

Further, let $f(\lambda)$ be an analytic function on G represented as a finite sum. Then for $u \in \mathrm{Exp}_{A,G}(\mathrm{X})$ with the representation $u = \sum_{\lambda \in G} u_\lambda$, $u_\lambda \in \mathrm{Exp}_{A,r,\lambda}(\mathrm{X})$, the operator $f(A)$ is defined by the formula

$$f(A)u = \sum_{\lambda \in G} f_\lambda(A) u_\lambda, \quad \text{where } f_\lambda(A) u_\lambda = \sum_{n=0}^{\infty} \frac{f^{(n)}(\lambda)}{n!} (A - \lambda I)^n u_\lambda. \tag{8}$$

In other words, each f_λ represents f locally in a neighborhood of $\lambda \in G$, and for u_λ the operator $f_\lambda(A)$ is well defined.

Additionally assume that there exists a one-parameter family of bounded invertible operators $U_\lambda :$ $X \to X$ such that

$$AU_\lambda - U_\lambda A = \lambda U_\lambda, \quad \lambda \in \sigma(A). \tag{9}$$

For example, if $X = L_2 \equiv L_2(R)$ and $A = -i\frac{d}{dx} : L_2 \to L_2$ with domain $\mathcal{D}(A) = \{v \in L_2\colon\ Av \in L_2\}$, then the operators $U_\lambda : v(x) \to e^{i\lambda x} v(x)$ satisfy

$$\begin{aligned} AU_\lambda v(x) &= -i\frac{d}{dx}\big(e^{i\lambda x} v(x)\big) = \lambda e^{i\lambda x} v(x) - i e^{i\lambda x} \frac{dv}{dx} \\ &= \lambda U_\lambda v(x) + U_\lambda A v(x), \end{aligned}$$

obtaining (9). Condition (9) indicates a shift of the spectrum of operator A to λ. This is seen from the relationship $A - \lambda I = U_\lambda A U_\lambda^{-1}$, which follows from (9) multiplying by U_λ^{-1} from the right. It follows from the latter that $(A - \lambda I)^n = U_\lambda A^n U_\lambda^{-1}$, for all $n = 1, 2, \dots$, yielding

$$f(A)u = \sum_{\lambda \in G} \sum_{n=0}^{\infty} \frac{f^{(n)}(\lambda)}{n!} U_\lambda A^n U_\lambda^{-1} u_\lambda.$$

Let X^* denote the dual of X, and $A^* : X^* \to X^*$ be the operator adjoint to A. Further, denote by $\mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$ the space of linear continuous functionals defined on $\mathrm{Exp}_{A,G}(\mathrm{X})$, with respect to weak convergence. Specifically, a sequence $u_m^* \in \mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$ converges to an element $u^* \in \mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$ if for all $v \in \mathrm{Exp}_{A,G}(\mathrm{X})$ the convergence $\langle u_m^* - u, v\rangle \to 0$ holds as $m \to \infty$. For an analytic function f^w defined on $G^* = \{z \in \mathbb{C}: \ \bar{z} \in G\}$, we define *a weak extension* of $f(A)$ as follows:

$$\langle f^w(A^*)u^*, v\rangle = \langle u^*, f(A)v\rangle, \quad \forall v \in \mathrm{Exp}_{A,G}(\mathrm{X}),$$

where $u^* \in \mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$.

Lemma 2.1. *Let X be a reflexive Banach space and A be a closed operator defined on $\mathcal{D}(A) \subset X$. Let f be an analytic function defined on an open connected set $G \subset \mathbb{C}$. Then the following mappings are well defined and continuous*:

1. $f(A) : \mathrm{Exp}_{A,G}(\mathrm{X}) \to \mathrm{Exp}_{A,G}(\mathrm{X})$,
2. $f^w(A^*) : \mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*) \to \mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$.

Proof. We will prove that $f(A)$ maps $\mathrm{Exp}_{A,G}(\mathrm{X})$ into itself. Let $u \in \mathrm{Exp}_{A,G}(\mathrm{X})$ have a representation $u = \sum_\lambda u_\lambda$, $u_\lambda \in \mathrm{Exp}_{A_\lambda,\nu}(\mathrm{X})$. Then for $f(A)u$ defined in (9), one has the following estimate

$$\left\| A_\lambda^k f_\lambda(A) u_\lambda \right\| \leqslant \sum_{n=0}^{\infty} \frac{|f^n(\lambda)|}{n!} \left\| (A - \lambda I)^n A_\lambda^k u_\lambda \right\| \leqslant C \nu^k \|u_\lambda\|, \tag{10}$$

with some $\nu < r$. It follows that $f_\lambda(A)u_\lambda \in \mathrm{Exp}_{A_\lambda,\nu}(\mathrm{X})$ with the same ν. Hence, $f(A)u$ has a representation $\sum_\lambda v_\lambda$, where $v_\lambda = f_\lambda(A)u_\lambda \in \mathrm{Exp}_{A_\lambda,\nu}(\mathrm{X})$, and therefore $f(A)u \in \mathrm{Exp}_{A,G}(\mathrm{X})$. The estimate (10) also implies continuity of the mapping $f(A)$ in the topology of $\mathrm{Exp}_{A,G}(\mathrm{X})$.

Now assume that a sequence $u_n^* \in \mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$ converges to 0 in the weak topology of $\mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$. Then for arbitrary $u \in \mathrm{Exp}_{A,G}(\mathrm{X})$ we have

$$\langle f^w(A^*)u_n^*, u\rangle = \langle u_n^*, f(A)u\rangle = \langle u_n^*, v\rangle,$$

where $v = f(A)u \in \mathrm{Exp}_{A,G}(\mathrm{X})$ due to the first part of the proof. Hence, $f^w(A^*)u_n^* \to 0$, as $n \to \infty$, in the weak topology of $\mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$. $\square$

Remark 2.2. Note that if $\sigma(A)$ is discrete then the space $\mathrm{Exp}_{A,G}(\mathrm{X})$ consists of the root lineals of eigenvectors corresponding to the part of $\sigma(A)$ with nonempty intersection with G. If the spectrum $\sigma(A)$ is empty, then an additional investigation is required for solution spaces to be non-trivial (for details see, [11]).

2.3. Two lemmas

The following two lemmas will be used in proofs of theorems in Section 3.

Lemma 2.3. *Let $h(t)$ be a continuously differentiable function for all $0 \leqslant t < T < \infty$. Then the equation $J^{\alpha} u(t) = h(t)$, $0 < t < T$, where $0 < \alpha < 1$, has a unique continuous solution given by the formula*

$$u(t) = D_{+}^{\alpha} h(t), \quad 0 < t < T. \tag{11}$$

Lemma 2.3 is essentially the well-known result on a solution of Abel's integral equation of first kind. Tonelli [40] sowed that if h is in a Hölder class $C^{\gamma}[0,T]$, $0 < \gamma < 1$, then a unique solution to $J^{\alpha} u(t) = h(t)$ is given by (11) and $u \in C^{\beta}[0,T]$ for some $\beta < \gamma - \alpha$. Note also, that if h is absolute integrable on $[0,T]$, then $u \in L_1(0,T)$. See [13,30] for further details.

Lemma 2.4. *Suppose $v(t,\tau)$ is an X-valued function defined for all $t \geqslant \tau \geqslant 0$, the derivatives $\frac{\partial^j v(t,\tau)}{\partial t^j}$, $0 \leqslant j \leqslant k-1$, are jointly continuous in the X-norm, and $\frac{\partial^k v(t,\tau)}{\partial t^k} \in L_1(0,t;X)$ for all $t > 0$. Let $u(t) = \int_0^t v(t,\tau)\,d\tau$. Then*

$$\frac{d^k}{dt^k} u(t) = \sum_{j=0}^{k-1} \frac{d^j}{dt^j} \left[\frac{\partial^{k-1-j}}{\partial t^{k-1-j}} v(t,\tau) \bigg|_{\tau=t} \right] + \int_0^t \frac{\partial^k}{\partial t^k} v(t,\tau)\,d\tau. \tag{12}$$

Proof. For a fixed $t > 0$ and small h one can easily verify that

$$\begin{aligned} \frac{u(t+h) - u(t)}{h} &= \frac{1}{h} \left(\int_0^{t+h} v(t+h,\tau)\,d\tau - \int_0^t v(t,\tau)\,d\tau \right) \\ &= \frac{1}{h} \int_t^{t+h} v(t+h,\tau)\,d\tau + \int_0^t \frac{v(t+h,\tau) - v(t,\tau)}{h}\,d\tau. \end{aligned} \tag{13}$$

Due to the continuity and differentiability conditions of the lemma, we have

$$\begin{aligned} \left\| \frac{1}{h} \int_t^{t+h} v(t+h,\tau)\,d\tau - v(t,t) \right\| &= \left\| \frac{1}{h} \int_t^{t+h} \big[v(t+h,\tau)d - v(t,t) \big]\,d\tau \right\| \\ &\leqslant \sup_{t<\tau<t+h} \big\| v(t+h,\tau) - v(t,t) \big\| = o(h), \quad h \to 0, \end{aligned} \tag{14}$$

$$\left\| \int_0^t \frac{v(t+h,\tau) - v(t,\tau)}{h}\,d\tau - \int_0^t \frac{\partial v(t,\tau)}{\partial t}\,d\tau \right\| = o(h), \quad h \to 0. \tag{15}$$

Now, letting $h \to 0$, estimates (14), (15) and Eq. (13) imply the formula

$$\frac{d}{dt} u(t) = v(t,t) + \int_0^t \frac{\partial}{\partial t} v(t,\tau)\,d\tau. \tag{16}$$

Formula (12) follows from (16) by repeated differentiation. $\square$

2.4. Classical Duhamel's principle

Let $B = B(x, \frac{\partial}{\partial t}, D_x)$, $D_x = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ be a linear differential operator with coefficients not depending on t, and containing temporal derivatives of order not higher than 1. Consider the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2}(t,x) + Bu(t,x) = h(t,x), \quad t > 0,\ x \in R^n, \tag{17}$$

with homogeneous initial conditions

$$u(0,x) = 0, \qquad \frac{\partial u}{\partial t}(0,x) = 0. \tag{18}$$

Let a sufficiently smooth function $v(t,\tau,x)$, $t \geqslant \tau$, $\tau \geqslant 0$, $x \in R^n$, be for $t > \tau$ a solution of the homogeneous equation

$$\frac{\partial^2 v}{\partial t^2}(t,\tau,x) + Bv(t,\tau,x) = 0,$$

satisfying the following conditions:

$$v(t,\tau,x)|_{t=\tau} = 0, \qquad \left.\frac{\partial v}{\partial t}(t,\tau,x)\right|_{t=\tau} = h(\tau,x).$$

Then a solution of the Cauchy problem (17), (18) is given by means of the integral

$$u(t,x) = \int_0^t v(t,\tau,x)\,d\tau. \tag{19}$$

The formulated statement is known as *Duhamel's principle*, and the integral in (19) as *Duhamel's integral.*

A similar statement is valid in the case of the Cauchy problem with a homogeneous initial condition for a first order inhomogeneous partial differential equation

$$\frac{\partial u}{\partial t}(t,x) + Cu(t,x) = h(t,x), \quad t > 0,\ x \in R^n,$$

where $C = C(x, D_x)$ is a linear differential operator containing only spatial derivatives, and with coefficients not depending on t (see [4]).

3. Generalizations of Duhamel's principle

3.1. Duhamel's principle for $\Lambda = \sum_{k=0}^m \delta_{\alpha_k}$ with $\alpha_k = k$, $k = 1, \ldots, m$

Suppose the measure Λ in (1) has the form $\Lambda = \sum_{k=0}^m \delta_k$, where δ_a denotes the Dirac delta function with mass at a. Suppose also that $f(m, A) = I$ is the identity operator. Then the Cauchy problem (1), (2) takes the form

$$u^{(m)}(t)+\sum_{k=0}^{m-1} f_k(A)u^{(k)}(t)=h(t), \quad t>0, \tag{20}$$

$$u^{(k)}(0)=\varphi_k, \quad k=0,\dots,m-1. \tag{21}$$

The operators $f_k(A)=f(k,A)$, $k=0,\dots,m-1$, are understood in the sense of the functional calculus introduced in Section 2.2. Duhamel's principle establishes a connection between the solutions of the Cauchy problem for nonhomogeneous equation (20) with the homogeneous initial conditions

$$u^{(k)}(0)=0, \quad k=0,\dots,m-1, \tag{22}$$

and the Cauchy problem for the corresponding homogeneous equation

$$\frac{\partial^m U}{\partial t^m}(t,\tau)+\sum_{k=0}^{m-1} f_k(A)\frac{\partial^k U}{\partial t^k}(t,\tau)=0, \quad t>\tau, \tag{23}$$

$$\frac{\partial^k U}{\partial t^k}(t,\tau)\bigg|_{t=\tau+0}=0, \quad k=0,\dots,m-2, \tag{24}$$

$$\frac{\partial^{m-1} U}{\partial t^{m-1}}(t,\tau)\bigg|_{t=\tau+0}=h(\tau). \tag{25}$$

Note that if $h(t)$ is a continuous $\mathrm{Exp}_{A,G}(\mathrm{X})$-valued ($\mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$-valued) function then the solution of (23)–(25) is an m times differentiable $\mathrm{Exp}_{A,G}(\mathrm{X})$-valued ($\mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$-valued) function (see [33]). Taking this fact into account, in the following theorem we assume that the vector-functions $h(t)$ and $U(t,\tau)$ are $\mathrm{Exp}_{A_\lambda,G}(\mathrm{X})$-, or $\mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$-valued, $h(t)$ is continuous, $U(t,\tau)$ is m times differentiable with respect to the variable t, and the derivatives $\frac{\partial^j U(t,\tau)}{\partial t^j}$, $0\leqslant j\leqslant k-1$, are jointly continuous in the topology of $\mathrm{Exp}_{A,G}(\mathrm{X})$, or of $\mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$, respectively. In the abstract case Duhamel's principle is formulated as follows.

Theorem 3.1. *Let $U(t,\tau)$ be a solution of the Cauchy problem* (23)–(25)*. Then a solution of the Cauchy problem* (20), (22) *is represented via Duhamel's integral*

$$u(t)=\int_0^t U(t,\tau)\,d\tau. \tag{26}$$

Proof. Let $u(t)$ be defined by (26). Obviously $u(0)=0$. Further, for the first order derivative of $u(t)$, using (12), one has

$$\frac{du}{dt}(t)=U(t,t)+\int_0^t \frac{\partial U}{\partial t}(t,\tau)\,d\tau.$$

By virtue of (24) the latter implies that $\frac{du}{dt}(0)=0$. Further, differentiating,

$$\frac{d^k u}{dt^k}(t)=\frac{\partial^{k-1} U}{\partial t^{k-1}}(t,t)+\int_0^t \frac{\partial^k U}{\partial t^k}(t,\tau)\,d\tau,$$

which due to condition (24) implies that $\frac{d^k u}{dt^k}(0) = 0$, $k = 2, \dots, m-1$. Therefore, the function $u(t)$ in (26) satisfies initial conditions (22). Moreover, substituting (26) to (20), and taking into account (25), we have

$$\begin{aligned}
u^{(m)}(t) + \sum_{k=0}^{m-1} f_k(A) u^{(k)}(t) &= \frac{d^m}{dt^m} \int_0^t U(t,\tau)\,d\tau + \sum_{k=0}^{m-1} f_k(A) \frac{d^k}{dt^k} \int_0^t U(t,\tau)\,d\tau \\
&= \frac{\partial^{m-1} U}{\partial t^{m-1}}(t,t) + \int_0^t \frac{\partial^m U}{\partial t^m}(t,\tau)\,d\tau + \sum_{k=0}^{m-1} f_k(A) \int_0^t \frac{\partial^k U}{\partial t^k}(t,\tau)\,d\tau \\
&= h(t) + \int_0^t \left[\frac{\partial^m U}{\partial t^m}(t,\tau) + \sum_{k=0}^{m-1} f_k(A) \frac{\partial^k U}{\partial t^k}(t,\tau) \right] d\tau = h(t).
\end{aligned}$$

Hence, $u(t)$ in (26) satisfies Eq. (20) as well. □

3.2. *Fractional Duhamel's principle for $\Lambda = \delta_\mu + \lambda$ with $\mu \in (m-1, m]$ and supp $\lambda \subset [0, m-1]$*

In this section we formulate fractional generalizations of Duhamel's principle and discuss some examples of their applications. Let $\Lambda = \delta_\mu + \lambda$, where μ is a number such that $m-1 < \mu < m$, and λ is a finite measure with *supp* $\lambda \subset [0, m-1]$. Consider the operator

$$ {}_\tau L^{(\mu,\lambda)}[u](t) \equiv {}_\tau D_*^\mu u(t) + \int_0^{m-1} f(\alpha, A)\, {}_\tau D_*^\alpha u(t)\, \lambda(d\alpha), \tag{27}$$

acting on m times differentiable vector-functions $u(t)$, $t \geqslant \tau \geqslant 0$. If $\tau = 0$, then instead of ${}_0L^{(\mu,\lambda)}[u](t)$ we write $L^{(\mu,\lambda)}[u](t)$.

Consider the Cauchy problem for the inhomogeneous equation

$$L^{(\mu,\lambda)}[u](t) = h(t), \quad t > 0, \tag{28}$$

with the homogeneous Cauchy conditions

$$u^{(k)}(0) = 0, \quad k = 0, \dots, m-1. \tag{29}$$

The fractional Duhamel principle establishes a connection between the solutions of this problem and the Cauchy problem for the homogeneous equation

$$ {}_\tau L^{(\mu,\lambda)}\big[V(\cdot,\tau)\big](t) = 0, \quad t > \tau, \tag{30}$$

$$\frac{\partial^k V}{\partial t^k}(t,\tau)\bigg|_{t=\tau+0} = 0, \quad k = 0, \dots, m-2, \tag{31}$$

$$\frac{\partial^{m-1} V}{\partial t^{m-1}}(t,\tau)\bigg|_{t=\tau+0} = D_+^{m-\mu} h(\tau), \tag{32}$$

where $h(t)$ is a given vector-function. In Theorem 3.2 we assume that the vector-functions $h(t)$, $t \geqslant 0$, and $V(t,\tau)$, $t \geqslant \tau \geqslant 0$, are $\mathrm{Exp}_{A,G}(\mathrm{X})$-, or $\mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$-valued, $h(t)$ is differentiable, $V(t,\tau)$ is an

m times differentiable with respect to the variable t, and the derivatives $\frac{\partial^j V(t,\tau)}{\partial t^j}$, $0 \leqslant j \leqslant k-1$, are jointly continuous in the topology of $\mathrm{Exp}_{A,G}(\mathrm{X})$, or of $\mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$, respectively.

Theorem 3.2. *Suppose that $V(t,\tau)$ is a solution of the Cauchy problem* (30)–(32)*. Then Duhamel's integral*

$$u(t) = \int_0^t V(t,\tau)\,d\tau \tag{33}$$

solves the Cauchy problem (28), (29)*.*

Proof. First notice that since $m-1<\mu<m$, or $0<m-\mu<1$, due to Lemma 2.3, the equation $J^{m-\mu}g(t) = h(t)$ has a unique solution

$$g(t) = D_+^{m-\mu}h(t). \tag{34}$$

Let $V(t,\tau)$ as a function of the variable t be a solution to Cauchy problem (30)–(32) for any fixed τ. We verify that $u(t) = \int_0^t V(t,\tau)\,d\tau$ satisfies Eq. (28), and conditions (29). Splitting the interval $(0, m-1]$ into subintervals $[0,1], (1,2], \ldots, (m-2, m-1]$, we have

$$L^{(\mu,\lambda)}[u](t) = D_*^{\mu}u(t) + \sum_{k=1}^{m-1} \int_{(k-1,k]} f(\alpha, A) D_*^{\alpha}u(t)\,\lambda(d\alpha). \tag{35}$$

For $\alpha \in (k-1,k)$, $k = 1, \ldots, m-1$, using the definition (3) of D_*^{α}, we have

$$D_*^{\alpha}u(t) = \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-s)^{k-\alpha-1} \frac{d^k}{ds^k} \int_0^s V(s,\tau)\,d\tau\,ds. \tag{36}$$

Lemma 2.4 and conditions (31) imply that

$$\frac{d^k}{ds^k} \int_0^s V(s,\tau)\,d\tau = \int_0^s \frac{\partial^k}{\partial s^k} V(s,\tau)\,d\tau, \quad k = 1, \ldots, m-1. \tag{37}$$

Hence,

$$D_*^{\alpha}u(t) = \int_0^t \frac{1}{\Gamma(k-\alpha)} \int_{\tau}^t (t-s)^{k-\alpha-1} \frac{\partial^k}{\partial s^k} V(s,\tau)\,ds\,d\tau. \tag{38}$$

Again due to Lemma 2.4 and condition (32),

$$\begin{aligned} \frac{d^m}{ds^m} \int_0^s V(s,\tau)\,d\tau &= \frac{\partial^{m-1}}{\partial s^{m-1}} V(s,\tau)\Big|_{\tau=s} + \int_0^s \frac{\partial^m}{\partial s^m} V(s,\tau)\,d\tau \\ &= D_+^{m-\mu}h(s) + \int_0^s \frac{\partial^m}{\partial s^m} V(s,\tau)\,d\tau. \end{aligned} \tag{39}$$

Therefore the first term on the right hand side of (35) takes the form

$$D_*^{\mu} u(t) = \frac{1}{\Gamma(m-\mu)} \int_0^t (t-s)^{m-\mu-1} \frac{d^m}{ds^m} \int_0^s V(s,\tau)\, d\tau\, ds$$
$$= \frac{1}{\Gamma(m-\mu)} \int_0^t (t-s)^{m-\mu-1} \left(D_+^{m-\mu} h(s) + \int_0^s \frac{\partial^m}{\partial s^m} V(s,\tau)\, d\tau \right) ds. \tag{40}$$

Furthermore, by virtue of (34),

$$\frac{1}{\Gamma(m-\mu)} \int_0^t (t-s)^{m-\mu-1} D_+^{m-\mu} h(s) = J^{m-\mu} D_+^{m-\mu} h(t) = h(t). \tag{41}$$

Now Eqs. (35), (38), (40), and (41) imply that

$$L^{(\mu,\lambda)}[u](t) = h(t) + \frac{1}{\Gamma(m-\mu)} \int_0^t (t-s)^{m-\mu-1} \int_0^s \frac{\partial^m}{\partial s^m} V(s,\tau)\, d\tau\, ds$$
$$+ \sum_{k=1}^{m-1} \int_{k-1}^{k} f(\alpha, A) \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-s)^{k-\alpha-1} \int_0^s \frac{\partial^k}{\partial s^k} V(s,\tau)\, d\tau\, ds\, \lambda(d\alpha). \tag{42}$$

Changing the order of integration (Fubini is allowed) in (42) we get

$$L^{(\mu,\lambda)}[u](t) = h(t) + \int_0^t \int_\tau^t \frac{1}{\Gamma(m-\mu)} (t-s)^{m-\mu-1} \frac{\partial^m}{\partial s^m} V(s,\tau)\, ds\, d\tau$$
$$+ \sum_{k=1}^{m-1} \int_0^t \int_{k-1}^{k} f(\alpha, A) \int_\tau^t \frac{1}{\Gamma(k-\alpha)} (t-s)^{k-\alpha-1} \frac{\partial^k}{\partial s^k} V(s,\tau)\, ds\, \lambda(d\alpha)\, d\tau$$
$$= h(t) + \int_0^t {}_\tau D_*^{\alpha_m} V(t,\tau)\, d\tau + \int_0^t \int_0^{m-1} f(\alpha, A)\, {}_\tau D_*^{\alpha} V(t,\tau)\, \lambda(d\alpha)\, d\tau$$
$$= h(t) + \int_0^t {}_\tau L^{(\mu,\lambda)}\big[V(\cdot,\tau)\big](t)\, d\tau = h(t).$$

Finally, using the relations (37) it is not hard to verify that $u(t)$ in (33) satisfies initial conditions (29) as well. □

If the vector-function h satisfies the additional condition $h(0) = 0$ then condition (32), in accordance with the relationship (5), can be replaced by

$$\left. \frac{\partial^{m-1} V}{\partial t^{m-1}}(t,\tau) \right|_{t=\tau} = D_*^{m-\mu} h(\tau),$$

with the Caputo–Djrbashian derivative $D_*^{m-\mu}$ of order $m-\mu$. As a consequence the formulation of the fractional Duhamel's principle takes the form:

Theorem 3.3. *Suppose that for all τ: $0<\tau<t$ a function $V(t,\tau)$ is a solution to the Cauchy problem for the homogeneous equation*

$$
\begin{aligned}
{}_\tau L^{(\mu,\lambda)}\big[V(\cdot,\tau)\big](t) &= 0, \quad t>\tau, \\
\frac{\partial^k V}{\partial t^k}(t,\tau)\bigg|_{t=\tau+0} &= 0, \quad k=0,\dots,m-2, \\
\frac{\partial^{m-1} V}{\partial t^{m-1}}(t,\tau)\bigg|_{t=\tau+0} &= D_*^{m-\mu}h(\tau),
\end{aligned}
$$

where $h(t)$ is a given differentiable vector-function such that $h(0)=0$. Then Duhamel's integral $u(t)=\int_0^t V(t,\tau)\,d\tau$ solves the Cauchy problem for the inhomogeneous equation (28), (29).

Remark 3.4.

1. Lemma 2.3 can be extended to absolutely continuous functions $h(t)$ with an appropriate meaning of solution in Eq. (11) (see, [30]). It is also known [30] that the fractional derivative $D_+^{k-\mu}h(t)$, $k-1<\mu<k$, $k=1,\dots,m$, exists a.e., if $h(t)$ is an absolutely continuous function on $[0,T]$ for $T>0$. These two facts imply that generalized Duhamel's principles proved above hold true for absolutely continuous functions $h(t)$.
2. In Theorems 3.2 and 3.3 we assumed that $f(\mu,A)$ is the identity operator (see Eq. (27)). In the general case, with appropriate selection of G, we can assume that the inverse operator $[f(\mu,A)]^{-1}$ exists. Then with the condition

$$
\frac{\partial^{m-1} V(t,\tau)}{\partial t^{m-1}}\bigg|_{t=\tau+} = \big[f(\mu,A)\big]^{-1} D_+^{m-\mu}h(\tau)
$$

   instead of (32), Theorems 3.2 and 3.3 remain valid.

3.3. Fractional Duhamels's principle with Riemann–Liuoville derivative

The operator ${}_\tau L^A$ in Theorem 3.2 is defined via the fractional derivative in the sense of Caputo and Djrbashian. A fractional generalization of Duhamel's principle is also possible when this operator is defined via the Riemann–Liouville fractional derivative. In this section we briefly discuss this important case proving the corresponding theorem in the simple particular case

$$
{}_\tau L[u](t) = {}_\tau D_+^\alpha u(t) + Bu(t), \tag{43}
$$

where $0<\alpha<1$, and B is a closed operator, independent of t, and with a domain $\mathcal{D}(B)$ dense in X. The initial value problem, called the Cauchy type problem, in this case has the form

$$
{}_\tau L[u](t) = h(t), \quad t>0, \tag{44}
$$

$$
{}_\tau J^{1-\alpha}u(\tau+) = \varphi \in X. \tag{45}
$$

The initial condition (45) can be rewritten as the weighted Cauchy type initial condition $\lim_{t\to\tau+}(t-\tau)^{1-\alpha}u(t)=\psi$ (see details in [18,30]).

Theorem 3.5. *Suppose that $V(t,\tau)$, $t \geqslant \tau \geqslant 0$, is a solution of the Cauchy type problem for the homogeneous equation*

$$ {}_{\tau}D_{+}^{\alpha}V(t,\tau) + BV(t,\tau) = 0, \quad t > \tau, \tag{46}$$

$$ {}_{\tau}J^{1-\alpha}V(t,\tau)\big|_{t=\tau+} = h(\tau), \tag{47}$$

where $0 < \alpha < 1$ and $h(\tau)$, $\tau \geqslant 0$, is a continuous vector-function. Then Duhamel's integral

$$u(t) = \int_0^t V(t,\tau)\,d\tau \tag{48}$$

solves the Cauchy type problem for the inhomogeneous equation

$$D_{+}^{\alpha}u(t) + Bu(t) = h(t), \quad t > 0, \tag{49}$$

with the homogeneous initial condition $J^{1-\alpha}u(0+) = 0$.

Proof. Let $V(t,\tau)$ satisfy the conditions of the theorem. Then for Duhamel's integral (48), by virtue of Lemma 2.4, we have

$$\begin{aligned} D_{+}^{\alpha}u(t) + Bu(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{\int_0^s V(s,\tau)\,d\tau}{(t-s)^{\alpha}}\,ds + \int_0^t BV(t,\tau)\,d\tau \\ &= \frac{d}{dt}\int_0^t {}_{\tau}J^{1-\alpha}V(t,\tau)\,d\tau + \int_0^t BV(t,\tau)\,d\tau \\ &= {}_{\tau}J^{1-\alpha}V(t,\tau)\big|_{\tau=t} + \int_0^t \big[{}_{\tau}D_{+}^{\alpha}V(t,\tau) + BV(t,\tau)\big]\,d\tau = h(t). \end{aligned} \tag{50}$$

On the other hand, changing the order of integration and using the mean value theorem, we obtain

$$\big\|J^{1-\alpha}u(t)\big\| = \left\|\int_0^t {}_{\tau}J^{1-\alpha}V(t,\tau)\,d\tau\right\| \leqslant t\big\|{}_{\tau_*}J^{1-\alpha}V(t,\tau_*)\big\|, \tag{51}$$

where $\tau_* \in (0,t)$, and the operator ${}_{\tau_*}J^{1-\alpha}$ on the rightmost term of (51) acts in the variable t. Condition (47) implies that $\lim_{t\to 0+} {}_{\tau}J^{1-\alpha}V(t,\tau) = h(0)$ in the norm of X. It follows from (51) that $\lim_{t\to 0+} J^{1-\alpha}u(t) = 0$ in the norm of X. $\square$

Remark 3.6. Theorem 3.5 can be generalized to differential-operator equations of the form

$$L_{+}^{\Lambda}[u] = D_{+}^{\mu}u(t) + \int_0^m B(\alpha)D_{+}^{\alpha}u(t)\,d\lambda(\alpha), \quad t > 0,$$

where $B(\alpha)$ is a family of closed operators, continuously depending on α, independent of the variable t, and with domains dense in X; the measure Λ is as in the definition of the operator ${}_\tau L^\Lambda$ in Eq. (27). The initial conditions of the reduced problem in this case, unlike the usual Cauchy conditions (31), (32) used in Section 3.2, have the form

$$
\begin{gathered}
{}_\tau D^{\alpha-j} V(t,\tau)\big|_{t=\tau+} = 0, \quad j = 1,\dots,m-1, \\
{}_\tau J^{m-\alpha} V(t,\tau)\big|_{t=\tau+} = h(\tau).
\end{gathered}
$$

Note that such initial conditions are natural for fractional order differential equations with the Riemann–Liouville fractional derivatives (see, e.g. [12,21]).

3.4. Applications. Existence and uniqueness theorems

Theorems 3.1–3.3 lead to generalization of the existence and uniqueness results obtained in papers [14,31] for the abstract Cauchy problems. Let L^Λ be the distributed fractional order abstract differential operator defined in (27) with $\tau = 0$, and via the characteristic function

$$
\Delta(s,z) = s^\mu + \int_0^{m-1} f(\alpha,z) s^\alpha \, d\lambda,
$$

where $\mu \in (m-1,m]$, λ is a finite measure with *supp* $\lambda \subset [0,m-1]$, and $f(\alpha,z)$ is a function continuous in α and analytic in $z \in G \subset \mathbb{C}$. Denote by $\hat{v}(s) = \mathcal{L}[v](s)$ the Laplace transform of a vector-function $v(t)$, namely

$$
\mathcal{L}[v](s) = \int_0^\infty e^{-st} v(t) \, dt, \quad s > s_0,
$$

where $s_0 \geqslant 0$ is a real number. It is not hard to verify that if $v(t) \in \mathrm{Exp}_{A,G}(X)$ for each $t \geqslant 0$ and satisfies the condition $\|v(t)\| \leqslant Ce^{\gamma t}$, $t \geqslant 0$, with some constants $C > 0$ and γ, then $\hat{v}(s)$ exists and

$$
\big\| A^k \hat{v}(s) \big\| \leqslant \frac{C_s}{s-\gamma} \nu^k, \quad s > \gamma,
$$

implying $\hat{v}(s) \in \mathrm{Exp}_{A,G}(X)$ for each fixed $s > \gamma$. The lemma below gives a formal representation formula for a solution of the general abstract Cauchy problem

$$
L^\Lambda[u](t) = h(t), \quad t > 0, \tag{52}
$$

$$
u^{(k)}(0+) = \varphi_k, \quad k = 0,\dots,m-1. \tag{53}
$$

Let $\delta_{j,k}$ denote the Kronecker delta, that is $\delta_{j,k} = 1$ if $j = k$, and $\delta_{j,k} = 0$, if $j \neq k$.

Lemma 3.7. *Let $c_\beta(t,z) = \mathcal{L}^{-1}[\frac{s^\beta}{\Delta(s,z)}](t)$, $z \in G \subset \mathbb{C}$, where $\mathcal{L}^{-1}$ stands for the inverse Laplace transform, and*

$$
S_k(t,z) = c_{\mu-k-1}(t,z) + \int_k^{m-1} f(\alpha,z) c_{\alpha-k-1}(t,z) \, \lambda(d\alpha), \quad k = 0,\dots,m-1. \tag{54}
$$

Then $S_k(t, A)\varphi_k$ solves the Cauchy problem

$$L^{\Lambda}[u] = 0, \qquad u^{(j)}(0) = \delta_{j,k}\varphi_j, \quad j = 0, \dots, m-1.$$

Proof. Applying formula (6) we have

$$\mathcal{L}\big[L^{\Lambda}[u]\big](s) = s^{\mu}\hat{u}(s) - \sum_{i=0}^{m-1} u^{i}(0)s^{\mu-i-1} + \sum_{k=1}^{m-1}\int_{k-1}^{k} f(\alpha, A)\left(s^{\alpha}\hat{u}(s) - \sum_{j=0}^{k-1} u^{j}(0)s^{\alpha-j-1}\right)\lambda(d\alpha) = 0.$$

Due to the initial conditions $u^{j}(0) = \delta_{j,k}\varphi_j$, $j = 0, \dots, m-1$, the latter reduces to

$$\Delta(s, z)\hat{u}(s) = \varphi_k\left(s^{\mu-k-1} + \int_{k}^{m-1} f(\alpha, z)s^{\alpha-k-1}\lambda(d\alpha)\right).$$

Now it is easy to see that the solution in this case is represented as $u_k = S_k(t, A)\varphi_k$, $k = 0, \dots, m-1$. $\square$

Corollary 3.8. *Let $S_k(t, A)$, $k = 0, \dots, m-1$, be the collection of solution operators with the symbols $S_k(t, z)$ defined in Lemma* 3.7. *Then the solution of the Cauchy problem*

$$L^{\Lambda}[u] = 0, \qquad u^{(j)}(0) = \varphi_j, \quad j = 0, \dots, m-1, \tag{55}$$

is given by the following representation formula

$$u(t) = \sum_{k=0}^{m-1} S_k(t, A)\varphi_k. \tag{56}$$

Remark 3.9.

1. Corollary 3.8 can easily be extended to the operator ${}_{\tau}L^{\Lambda}$ in (55) as well with the initial conditions $u^{j}(\tau) = \varphi_j$, $j = 0, \dots, m-1$. In this case the symbols of solution operators depend on τ and have the form $S_k(t, \tau, z) = S_k(t-\tau, z)$, $k = 0, \dots, m-1$, where $S_k(t, z)$ is defined in (54).
2. A particular case of Lemma 3.7 when $\Lambda = \sum_{k=0}^{m}\delta_{\alpha_k}$, $k-1 < \alpha_k < k$, is proved in [14].

Further, denote by $C^{(m)}[t > 0; \mathrm{Exp}_{A,G}(X)]$ and by $AC[t > 0; \mathrm{Exp}_{A,G}(X)]$ the space of m times continuously differentiable functions and the space of absolutely continuous functions on $(0; +\infty)$ with values ranging in the space $\mathrm{Exp}_{A,G}(X)$, respectively. A vector-function $u(t) \in C^{(m)}[t > 0; \mathrm{Exp}_{A,G}(X)] \cap C^{(m-1)}[t \geqslant 0; \mathrm{Exp}_{A,G}(X)]$ is called a solution of the problem (52), (53) if it satisfies Eq. (52) and the initial conditions (53) in the topology of $\mathrm{Exp}_{A,G}(X)$.

Theorem 3.2 and Corollary 3.8 imply the following results.

Theorem 3.10. *Let $\varphi_k \in \mathrm{Exp}_{A,G}(X)$, $k = 0, \ldots, m-1$, $h(t) \in AC[0 \leqslant t \leqslant T; \mathrm{Exp}_{A,G}(X)]$ for any $T > 0$, and $D_+^{m-\mu} h(t) \in C[0 \leqslant t \leqslant T; \mathrm{Exp}_{A,G}(X)]$. Then the Cauchy problem* (52), (53) *has a unique solution. This solution is given by*

$$u(t) = \sum_{k=0}^{m-1} S_k(t, A)\varphi_k + \int_0^t S_{m-1}(t-\tau, A) D_+^{m-\mu} h(\tau)\, d\tau. \tag{57}$$

Proof. We split the Cauchy problem (52), (53) into two Cauchy problems

$$L^{\Lambda}[U](t) = 0, \quad t > 0, \tag{58}$$

$$U^{(k)}(0+) = \varphi_k, \quad k = 0, \ldots, m-1, \tag{59}$$

and

$$L^{\Lambda}[v](t) = h(t), \quad t > 0, \tag{60}$$

$$v^{(k)}(0+) = 0, \quad k = 0, \ldots, m-1. \tag{61}$$

Due to Corollary 3.8 the unique solution to (58), (59) is given by

$$U(t) = \sum_{k=0}^{m-1} S_k(t, A)\varphi_k. \tag{62}$$

Lemma 2.1 implies that $U(t) \in C^{(m)}[t > 0; \mathrm{Exp}_{A,G}(X)]$. For the Cauchy problem (60), (61), in accordance with the fractional Duhamel's principle (Theorem 3.2), it suffices to solve the Cauchy problem for the homogeneous equation:

$${}_{\tau}L^{\Lambda}\big[V(t,\tau)\big](t) = 0, \quad t > \tau, \tag{63}$$

$$\left.\frac{\partial^k V(t,\tau)}{\partial t^k}\right|_{t=\tau+} = 0, \quad k = 0, \ldots, m-2, \tag{64}$$

$$\left.\frac{\partial^{m-1} V(t,\tau)}{\partial t^{m-1}}\right|_{t=\tau+} = D_+^{m-\mu} h(\tau). \tag{65}$$

The solution of this problem, again using Corollary 3.8 (with the note in Remark 3.9), has the representation

$$V(t,\tau) = S_{m-1}(t-\tau, A) D_+^{m-\mu} h(\tau). \tag{66}$$

Again it follows from Lemma 2.1 that $V(t,\tau) \in C^{(m)}[t > \tau; \mathrm{Exp}_{A,G}(X)]$ for all $\tau \geqslant 0$, as well as its Duhamel integral. Thus, Duhamel's integral of $V(t,\tau)$ and representation (62) lead to formula (57). The uniqueness of a solution also follows from the obtained representation (57) (see [33]). □

The duality immediately implies the following theorem.

Theorem 3.11. *Let* $\varphi_k^* \in \mathrm{Exp}'_{A^*,G^*}(X^*)$, $k = 0, \dots, m-1$, $h^*(t) \in AC[0 \leqslant t \leqslant T; \mathrm{Exp}'_{A^*,G^*}(X^*)]$ *and* $D_+^{m-\mu} h^*(t) \in C[t \leqslant t \leqslant T; \mathrm{Exp}'_{A^*,G^*}(X^*)]$. *Assume also that* $\mathrm{Exp}_{A,G}(X)$ *is dense in* X. *Then the Cauchy problem* (52), (53) (*with* A *switched to* A^*) *is meaningful and has a unique weak solution. This solution is given by*

$$u^*(t) = \sum_{k=0}^{m-1} S_k\big(t, A^*\big)\varphi_k^* + \int_0^t S_{m-1}\big(t-\tau, A^*\big) D_+^{m-\mu} h^*(\tau)\, d\tau.$$

Assume that $\mathrm{Exp}_{A,G}(X)$ is densely embedded into X. Besides, let the solution operators $S_k(t, A)$ for each $k = 0, \dots, m-1$, satisfy the estimates

$$\big\| S_k(t, A)\varphi \big\| \leqslant C\|\varphi\|, \quad \forall t \in [0, T], \tag{67}$$

where $\varphi \in \mathrm{Exp}_{A,G}(X)$, and $C > 0$ does not depend on φ. Then there exists a unique closure $\bar{S}_k(t)$ to X of the operator $S_k(t, A)$ which satisfies the estimate $\|\bar{S}_k(t)u\| \leqslant C\|u\|$ for all $u \in X$. Using the standard technique of closure (see [32,33]), we can prove the following theorem.

Theorem 3.12. *Let* $\varphi_k \in X$, $k = 0, \dots, m-1$, $h(t) \in AC[0 \leqslant t \leqslant T; X]$ *for any* $T > 0$, *and* $D_+^{m-\mu} h(t) \in C[0 \leqslant t \leqslant T; X]$. *Further let* $\mathrm{Exp}_{A,G}(X)$ *be densely embedded into* X, *and the estimates* (67) *hold for solution operators* $S_k(t, A)$, $k = 0, \dots, m-1$. *Then the Cauchy problem* (52), (53) *has a unique solution* $u(t) \in C^m[0 < t \leqslant T; X]$. *This solution is given by*

$$u(t) = \sum_{k=0}^{m-1} \bar{S}_k(t)\varphi_k + \int_0^t \bar{S}_{m-1}(t-\tau) D_+^{m-\mu} h(\tau)\, d\tau.$$

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